F. anal. 3.4.2025

- slides, info - nesto ake

A. Fast Fourier transform (FFT)

DFT of $F : \mathbb{Z}(N) \to \mathbb{C}$:

(1) $F(k) = \sum_{n=0}^{N-1} c_n^N \, \omega_N^{-kn}$, $c_n^N = \frac{1}{N} \sum_{k=0}^{N-1} F(k)\, \omega_N^{kn}$, $\omega_N = e^{-\frac{2\pi i}{N}}$

How to compute DFT, i.e. $c_0^N, \dots, c_{N-1}^N$, efficiently?
(= with fewest operations, (complex additions and multiplications)

Naive approach is $O(N^2)$:

Given $F(0), \dots, F(N-1)$, ω_N ; computing

(i) $\omega_N, \dots, \omega_N^{N-1}$ require $N-2$ mult.

(ii) each c_n^N require $N+1$ mult., $N-1$ additions

Total for $c_0^N, \dots, c_{N-1}^N$: $N-2 + N(N+1+N-1) = 2N^2 + N - 2 = O(N^2)$

FFT (Cooley-Tukey (Gauss)) is $O(N \log_2 N)$:

Take $N = 2^n$

Write $DFT(N) =$ " $\frac{1}{2}\left(DFT(\frac{N}{2}) + DFT(\frac{N}{2})\, \omega_N\right)$ "

Solve recursively

Let $F : \mathbb{Z}(2M) \to \mathbb{C}$ and def. $F_e, F_o : \mathbb{Z}(M) \to \mathbb{C}$ by

(2) $F_e(k) = F(2k)$, $\quad F_o(k) = F(2k+1)$

Key observation:

$$c_m^{2M}(F) = \frac{1}{2M} \sum_{k=0}^{2M-1} F(k)\, \omega_{2M}^{km}$$

$$= \frac{1}{2}\left[\frac{1}{M} \sum_{l=0}^{M-1} F(2l)\, \omega_{2M}^{(2l)m} + \frac{1}{M} \sum_{\tilde{l}=0}^{M-1} F(2\tilde{l}+1)\, \omega_{2M}^{(2\tilde{l}+1)m} \right]$$

$$\underset{\omega_{2M}^2 = \omega_M}{=} \frac{1}{2}\left[\frac{1}{M} \sum_{l=0}^{M-1} F_e(l)\, \omega_M^{lm} + \frac{1}{M} \sum_{\tilde{l}=0}^{M-1} F_o(\tilde{l})\, \omega_M^{\tilde{l}m} \cdot \omega_{2M} \right]$$

(3) $$= \frac{1}{2}\left[c_m^M(F_e) + c_m^M(F_o)\, \omega_{2M} \right]$$

FFT algorithm:

Computing $c_m^N(F)$ for $N = 2^n$:

$$\begin{tikzcd}[column sep=small]
 & \{c_m^{2^n}(F)\}_m \arrow[dl] \arrow[dr] & & & \\
\{c_m^{2^{n-1}}(F_e)\}_m \arrow[d] \arrow[dr] & & \{c_m^{2^{n-1}}(F_o)\}_m \arrow[drr, dotted] & & \\
\{c_m^{2^{n-2}}(F_{ee})\}_m \arrow[d, dotted] & \{c_m^{2^{n-2}}(F_{eo})\}_m & & & \\
\{c_m^2(F_{e\cdots e})\}_m & \{c_m^2(F_{e\cdots eo})\}_m & \cdots & & \{c_m^2(F_{o\cdots o})\}_m
\end{tikzcd}$$

induction ↑ recursion ↓

Obs: $$c_m^2(G) = \frac{1}{2}\left(G(0)\,\omega_2^{0\cdot m} + G(1)\,\omega_2^{1\cdot m}\right)$$

(4) $$= \frac{1}{2}\left(G(0) + (-1)^m G(1)\right), \quad m = 0, 1$$

Operations count:

$\#(M)$ = no. of operations to compute DFT on $\mathbb{Z}(M)$

Lem. 1: Given F and $\omega_{2M} = e^{-\frac{2\pi i}{2M}}$, then

$$\#(2M) \leq 2 \cdot \#(M) + 8M$$

Pf.:

(i) computing $\omega_{2M}, \dots, \omega_{2M}^{2M-1}$ $\leq 2M$ mult'ns

(ii) computing $\{c_m^M(F_e)\}_m$, $\{c_m^M(F_o)\}_m$

$= 2 \cdot \#(M)$

(iii) computing $\{c_m^{2M}(F)\}_m$ from (3)

$$= \underbrace{2 \cdot \#(M)}_{(ii)} + \underbrace{2M \cdot (2 \text{ mult'ns} + 1 \text{ addition})}_{(3)} + \underbrace{2M}_{(i)}$$

$$= 2 \cdot \#(M) + 8M \qquad \square$$

Thm. 1: Let $N = 2^n$ and F, ω_{2M} be given.

Then FFT requires

$$\#(N) \leq 4 \cdot 2^n \cdot n = 4N \lg_2 N = O(N \log N) \text{ op'ns} \qquad (5)$$

Pf.:

Induction:

$n = 1$: By (4), $\#(1) \leq 2 \cdot (1 \text{ addition} + 2 \text{ mult'ns}) =$

$\leq 4 \cdot 2^1 \cdot 1 = 8$ op'ns

so (5) holds

Assume (5) holds for $N = 2^{n-1}$:

By Lem 1,

$$\#(2^n) \leq 2 \cdot \#(2^{n-1}) + 8 \cdot 2^{n-1}$$

$$\overset{(5)\text{ for }2^{n-1}}{\leq} 2 \cdot \left(4 \cdot 2^{n-1} \cdot (n-1)\right) + 4 \cdot 2^n$$

$$= 4 \cdot 2^n \cdot n - 4 \cdot 2^n + 4 \cdot 2^n$$

$\Longrightarrow$ (5) holds w. $N = 2^n$ $\square$

SS 7.3 Exercises:

(1) $f \in C(\mathbb{S}) \simeq C_{per}(\mathbb{R})$ (1-per. cont. f'ns on $\mathbb{R}$)

Obs: $F(k) := f(e^{2\pi i \frac{k}{N}}) \Rightarrow F \in C(\mathbb{S}_N) \simeq C(\mathbb{Z}(N))$
$$\simeq C_{N\text{-per}}(\mathbb{Z})$$

DFT of f:

$$c_n^N = \frac{1}{N} \sum_{k=0}^{N-1} f(e^{2\pi i \frac{k}{N}}) e^{-2\pi i n \frac{k}{N}}$$

F.-coeff of f:

$$c_n = \int_0^1 f(e^{2\pi i x}) e^{-2\pi i n x} \, dx$$

(a) $c_{n+N}^N = \frac{1}{N} \sum_k f(e^{2\pi i \frac{k}{N}}) \underbrace{e^{-2\pi i (n+N)\frac{k}{N}}}_{= e^{-2\pi i n \frac{k}{N}} \cdot \underbrace{e^{-2\pi i k}}_{=1}} = c_n^N$

(b) $c_n^N = \sum_{k=0}^{N-1} \underbrace{f(e^{2\pi i x_k}) e^{-2\pi i n x_k}}_{\text{cont. in } x_k,} \Delta x$, $x_k = \frac{k}{N}$, $\Delta x = x_{k+1} - x_k = \frac{1}{N}$

$\Downarrow$ $\{x_k\}_k$ partition of $[0,1]$

$N \to \infty \Rightarrow \Delta x \to 0$

Classical results for R.-integrals

$$\lim_{N \to \infty} c_n^N = \int_0^1 f(e^{2\pi i x}) e^{-2\pi i n x} \, dx = c_n$$

② $f \in C^1(\mathbb{S})$. Show $|c_n^N| \le \frac{C}{n}$, $0<|n| \le \frac{N}{2}$

Obs: $F : \mathbb{Z}(N) \to \mathbb{C}$, then

$$\sum_{k=l}^{N-1+l} \underbrace{F(k)}_{\substack{\shortparallel \\ F(k \bmod N)}} = \sum_{k=0}^{N-1} F((k+l) \bmod N) = \sum_{m=0}^{N-1} F(m)$$

Then

$$c_n^N (1 - e^{2\pi i n \frac{l}{N}}) = \frac{1}{N} \sum_k f(e^{2\pi i \frac{k}{N}})(e^{-2\pi i n \frac{k}{N}} - e^{-2\pi i n \frac{k-l}{N}})$$

$$\overset{m=k-l}{=} \frac{1}{N} \sum_k (\cdots) + \underbrace{\frac{1}{N} \sum_{m=-l}^{N-l-l} \underbrace{f(e^{2\pi i \frac{m+l}{N}}) e^{-2\pi i n \frac{m}{N}}}_{=:G : \mathbb{Z}(N) \to \mathbb{C}}}_{\overset{\text{Obs}}{=} \sum_{m=0}^{N-1} f(e^{2\pi i \frac{m+l}{N}}) e^{-2\pi i n \frac{m}{N}}}$$

$$\overset{k=m}{=} \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{(f(e^{2\pi i \frac{k}{N}}) - f(e^{2\pi i \frac{k+l}{N}}))}_{\overset{\text{Taylor}}{=} f'(e^{2\pi i \frac{\xi}{N}}) \underbrace{(e^{2\pi i \frac{k+l}{N}} - e^{2\pi i \frac{k}{N}})}_{= e^{2\pi i \frac{k}{N}} \underbrace{(e^{2\pi i \frac{l}{N}} - 1)}_{\overset{\text{Taylor}}{=} (e^x)'_{x = 2\pi i \frac{\tilde{\xi}}{N}} \cdot \frac{2\pi i l}{N},\ \tilde{\xi} \in (0,l)}},\ \xi \in (k,k+1)} e^{-2\pi i n \frac{k}{N}}$$

Hence

$$(*) \quad |c_n^N| \cdot |1 - e^{2\pi i n \frac{l}{N}}| \le \frac{1}{N} \sum_{n=0}^{N-1} \|f'\|_\infty \cdot 1 \cdot 1 \cdot \frac{2\pi l}{N} = 2\pi \|f'\|_\infty \frac{l}{N}$$

For every n, $0 < |n| \leq \frac{N}{2}$, take $l = l(n)$ s.t.

$$\left| \frac{nl}{N} - \frac{1}{2} \right| < \frac{1}{N} \qquad \text{(always ok!)}$$

Then

$$\underbrace{|1 - e^{2\pi i \frac{nl}{N}}| \geq}_{\Delta\text{-ineq.}} \underbrace{|1 - e^{\pi i}|}_{=2} - \underbrace{|e^{\pi i} - e^{2\pi i \frac{nl}{N}}|}_{\overset{\text{Taylor}}{=} (e^z)'_{z=i\eta} \cdot (2\pi i \frac{nl}{N} - \pi)}$$

$$\geq 2 - 1 \cdot 2\pi \frac{1}{N} \quad (> 0 \text{ if } N \geq 4)$$

and

$$\frac{l}{N} \leq \frac{1}{n} \cdot \frac{1}{2} + \frac{1}{n} \left| \frac{nl}{N} - \frac{1}{2} \right| < \frac{1}{n} \left(\frac{1}{2} + \frac{1}{N} \right) \leq \frac{3}{2n}$$

Conclusion: For $N \geq 4$ and $0 < |n| < \frac{N}{2}$,

$$(*) \Rightarrow |c_n^N| \leq \frac{\cancel{2}\pi \frac{3}{\cancel{2}n}}{\cancel{2}\underbrace{(1 - \frac{\pi}{4})}_{\geq \frac{1}{2}}} \|f'\|_\infty \leq \frac{3\pi \|f'\|_\infty}{n}$$

[Optional]

3 (a) $f \in C^2(\mathbb{S}) \Rightarrow |c_n^N| \le \frac{C}{n^2}, \quad 0 < |n| \le \frac{N}{2}$

As in 2:

$$c_n^N \left(e^{2\pi i n \frac{l}{N}} - 2 + e^{-2\pi i n \frac{l}{N}} \right)$$

$$= \text{as in } ② = \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{\left(f(e^{2\pi i \frac{k+l}{N}}) - 2f(e^{2\pi i \frac{k}{N}}) + f(e^{2\pi i \frac{k-l}{N}}) \right)} \cdot e^{-2\pi i n \frac{k}{N}}$$

$$|c_n^N| \cdot |e^{-} - 2 - e^{--}|$$

Taylor

$$\le \frac{1}{N} \sum_k \|f''\|_\infty \left| e^{2\pi i \frac{k}{N}} (e^{\pm 2\pi i \frac{l}{N}} - 1) \right|^2 \cdot \left| e^{-2\pi i n \frac{k}{N}} \right|$$

$$\le \|f''\|_\infty \, 2\pi \left(\frac{l}{N} \right)^2$$

as in 2

$$\le \frac{C \|f''\|_\infty}{n^2}$$

(b) Inversion:

Let N be odd, and note (by Obs in 2) that

$$(**) \quad f(e^{2\pi i \frac{k}{N}}) = \sum_{k=0}^{N-1} c_n^N e^{2\pi i n \frac{k}{N}} = \sum_{|k| < \frac{N}{2}} c_n^N e^{2\pi i n \frac{k}{N}}, \quad k \in \mathbb{Z}(N)$$

Obs: $c_n^N \longrightarrow c_n$ by 1

$|c_n^N| \le \frac{C}{n^2}$ by (a)

Take $x_m = \dfrac{k(m)}{\underset{\uparrow \text{odd}}{N(m)}} \longrightarrow x \in [0,1)$ and $N(m) \to \infty$. By (**)

$$
\begin{tikzcd}
f(e^{2\pi i x_m}) \arrow[d, "f \text{ cont.}"', "m\to\infty"] & = & \displaystyle\sum_{|k|<\frac{N}{2}} \arrow[d, "m\to\infty"'] & c_n^N \arrow[d, "m\to\infty \Rightarrow N\to\infty"'] & e^{2\pi i x_m} \arrow[d] \\
f(e^{2\pi i x}) & & & c_n & e^{2\pi i x} \text{ uniformly} \\
& & \displaystyle\sum_{k\in\mathbb{Z}} c_n e^{2\pi i x} \arrow[u, phantom, "\text{series abs. and unif conv. by (a)}"] & &
\end{tikzcd}
$$

Hence

$$f(e^{2\pi i x}) = \sum_{k\in\mathbb{Z}} c_n e^{2\pi i x}$$